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by Arthur Bowes Frizell

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THE FOUNDATIONS OF ARITHMETIC.

DISSERTATION FOR THE DEGREE OF DOCTOR OF PHILOSOPHY, SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL OF THE UNIVERSITY OF KANSAS.

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This thesis seeks a foundation for arithmetic in the ideas underlying Cantor's formulation of his system of ordinal types.

It proceeds by postulating, but follows D. Hilbert and G. Peano rather than E. V. Huntington. The search is for postulates possessing heuristic and didactic, not merely subsumptive value.

A *motif* is found in the notion of an abstract group, which secures the development step by step of all number systems so far studied without further postulates than those needed for the transfinite ordinals.

As axioms are to be avoided, it is necessary to state carefully the definitions and theorems used even when they are well known to the mathematicians, but in this case the proofs are omitted.

1. *Definition.* A set of symbols $a, b, \ldots$ will be said to form a K-class if we possess a test which enables us to assert in every case either that $a = b$ or that a is not $= b$ subject only to the restrictions that

a) the result of comparison be uniquely determined
b) the statements $b = a$ and $a = b$ shall be interchangeable
c) from $a = b$ and $b = c$ must follow $a = c$
d) $a = a$ if K contains no other symbol equal to a.

2. *Definition.* An infinite set of symbols is one which contains parts that can be put into one correspondence with the whole. A finite set is one that is not infinite.

3. *Definition.* An ordered set is one in which we are always able to say either that a precedes b ($a < b$) or that a follows b ($a > b$) subject only to the restrictions that
a) the result of comparison is uniquely determined
b) $a < b$ shall exclude $a = b$ but involve $b > a$
c) if $a < b$ and $b < c$, then $a < c$
d) if $a' = a$ and $b' = b$, then $a' < b'$ or $a' > b'$ according as $a < b$ or $a > b$.

4. *Definition.* An ordered set is said to be well ordered when it has a first element and every subset beginning with the first element has an immediate successor in the given set.

5. *Definition.* A procedure whereby to every a, b of a K-class is assigned a definite symbol $c = a \circ b$ will be called a C-rule or rule of combination provided that by this assignment equals with equals give equals.

6. *Definition.* A K-class is said to possess the fundamental group property with respect to a C-rule when $a \circ b$ also belongs to K.

7. *Definition.* Modulus of a K-class with regard to a C-rule is a symbol u of K such that $a \circ u = a = u \circ a$ for every a in K.

8. *Proposition I.* An ordered set defined by the requirements: **a**) it shall contain a given symbol e; **b**) it shall possess the group property for every combi-

nation $e \circ a$ where a denotes a previously defined member of the set is also well ordered, infinite and forms a K-class with respect to the given rule. *Proof:* By hypothesis the set is to contain $e \circ e = e'$, $e \circ e' = e''$, $e \circ e'' = e'''$, . . . Since it is to be an ordered set no two of its members can be equal $\therefore$ to every element of the whole set can be assigned some one of the subset e', e''', e^{v}, . . . That is, the set is infinite. And by **b**) every element has an immediate successor. Finally we have a test for equality which satisfies the requirements of § 1.

9. *Scholium.* A set defined as in § 8 contains no modulus. In what follows it will be referred to as an e-set.

10. *Postulates.* We postulate an e-set for a lower rule of combination which shall contain a lower symbol u and e-sets for both this and a higher rule which shall both contain a higher symbol w, the definitions of the rules of combination to be completed in §§ 17, 26, 28, 30.

11. The different sets $K[u \circ]$, $K[w \circ]$, $K[w \square]$ are to be so ordered that the lower shall precede the higher. Thus they form together a well ordered set

$$u, \; u \circ u = u', \; u \circ u' = u'', \; u \circ u'' = u''', \; \dots .$$
$$w, \; w \circ w = wu', \; w \circ wu' = wu'', \; w \circ wu'' = wu''', \; \dots$$
$$w \square w = w^{u'}, \; w \square w^{u'} = w^{u''}, \; w \square w^{u''} = w^{u'''}, \; \dots$$

12. *Postulates.* We postulate e-sets for the lower rule which shall contain respectively each of the symbols $w^{u'}$, $w^{u''}$, $w^{u'''}$, . . . in succession. Thus corresponding to every a of the set $K[w \square]$ we shall have an e-set $K[a \circ] : a$, $a \circ a = au'$, $a \circ au' = au''$, . . . Now taking *e. g.* $a = w^{u'''}$, by the principle of § 11 every member of $K[w^{u'''} \circ]$ precedes $ww^{u'''} = w^{u^{iv}}$ and therefore falls between $w^{u'''}$ and $w^{u^{iv}}$. That is, all these new e-sets are interpolated between successive

elements of $K[w\square]$. Hence the totality of symbols forms a well ordered set.

13. *Proposition II.* A rule of combination is associative throughout a given K-class if

$$a\circ(b\circ c)=a\circ b\circ c$$

for every a, b, c in K.

14. *Definition.* A K-class is said to form a semigroup with regard to a C-rule associative throughout K if by this rule unequals with equals always give unequals or if from the relation $a'\circ b=a\circ b$, resp. $a\circ b'=a\circ b$ we can always infer $a'=a$ resp. $b'=b$.

15. *Proposition III.* No semigroup contains more than one modulus for its defining C-rule. For if u and u' not $=u$ were both moduli, we should have $a\circ u'=a=a\circ u$ contrary to § 14.

16. *Definition.* An abelian K-class with reference to a C-rule is one for which the rule is without exception commutative.

17. *Proposition IV.* Sufficient conditions of an abelian class are the relations

$$a\circ(b\circ c)=a\circ b\circ c \quad \text{and} \quad b\circ a=a\circ b$$

for every a, b, c in the class.

18. *Proposition V.* A C-rule is distributive over another in a given K-class if the relations

$$(a\circ b)c=ac\circ bc \quad \text{and} \quad a(b\circ c)=ab\circ ac$$

are satisfied by every a, b, c in K.

19. *Proposition VI.* A necessary and sufficient condition of the generating rule of an e-set being associative for the set is the inductive formula of definition.

$$e\circ a\circ b=e\circ(a\circ b).$$

Proof. If the relation of $a\circ(b\circ c)=a\circ b\circ c$ has been established for a certain a and every b, c of the set, then by hypothesis $e\circ a\circ(b\circ c)=e\circ\{a\circ(b\circ c)\}=e\circ\{a\circ b\circ c\}=e\circ(a\circ b)\circ c=e\circ a\circ b\circ c$.

But $e \circ b \circ c = e \circ (b \circ c)$ by definition.

Therefore $e' \circ b \circ c = e' \circ (b \circ c)$ and so on.

Hence the theorem by strict induction.

20. *Corollary.* The rule is also commutative. For if $e \circ a = a \circ e$, then by the associative law $e \circ a \circ e = e \circ (a \circ e) = e \circ (e \circ a)$. But $e \circ e' = e' \circ e$ by § 19, therefore $e \circ e'' = e'' \circ e$, and so on. And if $b \circ a = a \circ b$ for a certain a and every b then $b \circ (e \circ a) = b \circ (a \circ e) = b \circ a \circ e = a \circ b \circ e = e \circ (a \circ b) = e \circ a \circ b$.

Hence the proposition follows by strict induction.

21. *Proposition VII.* The formula of § 19 is also a sufficient condition of the e-set constituting an abelian semigroup for its generating C-rule. *Proof.* If $a \circ b$ belongs to the given e-set, then $e \circ a \circ b = e \circ (a \circ b)$ also belongs to it. Thus the first group property is established. We have proved the associative law and by definition no two numbers of the set are equal. Therefore $a' \circ b$ is not $= a \circ b$ when a' is not $= a$.

22. *Postulates.* Using capital letters M, N, ... to denote elements of the lowest class $K[u \circ]$, and italic letters a, b, . . . for those of the class $K[w \square]$ where a shall precede b, we postulate for every $MB = BM$ a set $MB \circ Na$ for every Na, then using letters a, b, ... where a shall precede b to denote the resulting symbols we postulate inductively new sets b ∘ a where a $= Na$ and b denotes successively the higher "polynomial" or composite symbols. Every b-set is to be simply ordered and possess the first group property for every b ∘ a.

23. *Proposition VIII.* Every b-set is well ordered, infinite and forms a K-class for the rule denoted by ∘. For the b-set consists of the combinations b, b ∘ a, b ∘ a u', b ∘ a u'', b ∘ a u''', ... and the reasoning of *Prop. I* holds good.

24. *Proposition IX.* The b-sets belonging to a

given $M b$ form together an infinite, well ordered K-class as regards the rule $\circ$. For the set of b's:

$$M b, \; M b \circ w^{(n)}, \; M b \circ w^{(n \circ u)}, \; \ldots .$$

forms a well ordered K-class between consecutive members of which the b-sets are interpolated.

25. *Proposition X.* The totality of the b-sets forms an infinite, well ordered K-class as regards the rule $\circ$. For the set of $M b$ belonging to a given b is a well ordered K-class and so is the whole set of b's, the latter being simply K[$w \square$].

26. New combinations of symbols already defined are defined inductively in accordance with the formulas

$$b \circ a = a \circ b \qquad \text{and}$$
$$a \circ (b \circ c) = a \circ b \circ c.$$

This secures the group property for the whole set and the reasoning of *VI* and *VII* establishes

Proposition XI. The set of symbols generated according to two rules, a higher and a lower, from a single arbitrary symbol w by aid of the postulate of order, the restricted form of the group property and the above inducted formulas of definition, is a well ordered, infinite set constituting an abelian semigroup with regard to the lower rule.

27. *Scholium.* The higher rule remains unrestricted and has not been defined beyond the set K[$w \square$], therefore is not yet a C-rule for the K-class of § 26.

28. *Definition.* The higher rule shall be defined inductively throughout K[$w \square$] by the formula

$$w\, a\, b = w \square a\, b,$$

where a, b are any members of K[$w \square$] for which $a b$ has already been defined.

29. *Proposition XII.* The class K[$w \square$] forms an abelian semigroup with regard to its generating rule. Proof by *Propositions VI* and *VII.*

30. *Definition.* The higher rule shall be defined inductively for the remaining symbols of our abelian semigroup on the lower rule according to the formulas

$$(a \circ b)c = ac \circ bc \text{ and } a(b \circ c) = ab \circ ac,$$

where the combinations ab, ac, bc shall have been already defined.

31. *Proposition XIII.* The higher rule is distributive over the lower throughout the class of symbols defined in § 22. *Proof.* First let $a = b = c = l$ where l is any symbol of $\mathrm{K}[w \,\square]$. Then ll is defined by § 28 and $l(l \circ l) = ll \circ ll = (l \circ l)l$ by § 30. Now suppose that the formulas of § 30 have been established for all symbols of $\mathrm{K}[l \circ]$ up to and including a certain a and for every b and c.

$$\begin{aligned} \text{Then } (l \circ a \circ b)c &= \{l \circ (a \circ b)\}c && \text{by associative law,} \\ &= lc \circ (a \circ b)c && \text{by hypothesis,} \\ &= lc \circ (ac \circ bc) && \text{by hypothesis,} \\ &= lc \circ ac \circ bc && \text{by associative law,} \\ &= (l \circ a)c \circ bc. \end{aligned}$$

Similarly in every other case when a, b, c are replaced by $l \circ a$, $l \circ b$, $l \circ c$ respectively. But by definition $l(l \circ \mathrm{N}a) = ll \circ a \,\square\, \mathrm{N}a$ and so on. Therefore by strict induction the formulas hold for every a, b, c belonging to the same $\mathrm{K}[l \circ]$. Hence by *Prop. V* the theorem is true in this case. If h, k, l denote different members of $\mathrm{K}[w \,\square]$ the definition gives

$$(h \circ k)l = hl \circ kl \text{ and } h(k \circ l) = hk \circ hl.$$

Then the proof is completed inductively by the same reasoning as above.

32. *Corollary 1.* The higher rule is a C-rule and by it unequals with equals give unequals. For since our symbols form a well ordered set according to the lower rule, we have $\mathrm{a} = \mathrm{b} \circ \mathrm{x}$ for any two unequal elements a, b and therefore $\mathrm{ac} = \mathrm{bc} \circ \mathrm{xc}$ ∴ not $= \mathrm{bc}$.

33. *Corollary 2.* The higher rule is associative.

For this property has been established for the elements a, b, ... as members of K[w □]. Therefore by the distributive law it holds generally.

34. *Corollary 3.* The higher rule is commutative. For this also has been proved in the class K[w □].

35. *Corollary 4.* The set of symbols defined in § 22 forms an abelian semigroup with reference to the higher rule.

36. *Scholium.* Beginning with any $a > w$ we have abelian semigroups with regard to both rules, and so beginning with w for one or the other rule, but no part beginning with w forms a semigroup on both rules.

37. *Proposition XIV.* Necessary and sufficient conditions that a set M constitute an abelian semigroup with respect to each of two rules of combination, one distributive over the other, that M contain an arbitrary w and that M be as generated an ordered set, are the rules of combination thus far defined.

38. *Corollary.* The properties in question may be expressed by the following postulates:

A. There shall be a higher and a lower rule of combination.

B. Combinations of symbols shall be so ordered that those made by the lower rule precede combinations of the same symbols by the higher.

C. The higher rule shall be distributive over the lower.

D. There shall be a set M containing the symbol w.

E. M shall form an abelian semigroup for the lower rule.

F. M shall form an abelian semigroup for the higher rule.

G. Every successive set generated according to requirements shall be an ordered set.

39. *Scholium.* M is a well ordered set with no modulus.

40. We postulated two rules of combination for the symbol w, but only one for the lower symbol u. If we should postulate a higher rule for u by the same requirements as for w the resulting set of symbols would not differ as regards any group property from the w-set; the new abelian semigroups would be respectively holoedrically isomorphic with the former. If we postulate no higher rule the set $K[u\circ]$ still forms an abelian semigroup holoedrically isomorphic with $K[w\circ]$.

41. Instead let us set up postulates for the u class of symbols differing from those of § 38 only in adding another postulate H. The symbol u shall be a modulus for the higher rule, with the restriction on A–G that they shall not conflict with H. Then the class $K[u\square]$ reduces to the single element u, the abelian semigroup on the two rules reduces to the set $K[u\circ]$ and postulate B drops out, since every combination by the higher rule is found among those made according to the lower rule. This new set of postulates may be replaced by the equivalent set.

42. **a.** There shall be a higher rule defined from a lower by the inductive formulas $(u\circ a)b = ub\circ ab$ and $a(u\circ b) = au\circ ab$.

b. There shall be a lower rule defined inductively by the formula $u\circ a\circ b = u\circ(a\circ b)$.

c. There shall be a set M containing the symbol u.

d. The set M shall possess the group property for every combination $u\circ a$.

e. M shall be an ordered set.

f. There shall be a set M′ built up by postulates c, d and e on the symbol uu.

g. The sets M′ and M shall be identical.

43. *Proposition XV.* The set of symbols defined by the above seven postulates constitutes an abelian semigroup for each rule and has a modulus for the higher rule, which is distributive over the lower.

Proof. The abelian semigroup on the lower rule is established by *Prop. VII*, the distributive law follows from XIII and the other semigroup by § 35. It remains to prove the existence of a modulus.

Let $e = uu$. By postulate g, e must occur in the set $K[u\circ] : u, u', u'', \ldots$

Suppose $e = u''$. Then every member of $K[e\circ]$ will be found in $K[u\circ]$ but beyond u'. That is, u and u' are not in $K[e\circ]$, contrary to g. Therefore the only possibility of satisfying g is $e = u$. Since, then, $uu = u$, it follows by the distributive law that u is a modulus.

44. Postulates a . . . g define the natural numbers as ordinal symbols and by *XV* contain all the laws of their arithmetic. The cardinal numbers may be defined in a manner now familiar as names of classes, *e. g.*, "five" is the name given to the class of all well ordered sets which are ordinally similar to the set u, u', u'', u''', u^{iv}.

45. *Definition.* A group is a semigroup which contains, corresponding to every a, b, in it, symbols p and q such that $a\circ p = b = q\circ a$.

46. *Proposition XVI.* Every group contains a modulus with respect to its defining C-rule.

47. *Definition.* If a class which has a modulus u for its defining C-rule contains symbols a, $\bar{a}$ such that $a\circ\bar{a} = u = \bar{a}\circ a$ then a, $\bar{a}$ are said to be each the *inverse* of the other.

48. *Proposition XVII.* No member of a semigroup can have more than one inverse in the semigroup. For otherwise equals with unequals would give equals.

49. *Proposition XVIII.* Every group contains the inverse of every one of its members.

50. *Proposition XIX.* A semigroup with modulus is a group if it also contains the inverse of every one of its members.

Proof. $a \circ (\bar{a} \circ b) = a \circ \bar{a} \circ b = u \circ b = b$, and $(b \circ \bar{a}) \circ a = b \circ (\bar{a} \circ a) = b \circ u = b$, that is, § 45 is satisfied by $p = \bar{a} \circ b$ and $q = b \circ \bar{a}$.

51. *Proposition XX.* Given an abelian semigroup G with reference to a rule denoted by $\circ$, if in the class $[(m, n)] =$ C of pairs of elements of G we declare $(m, q) = (n, p)$ when and only when $m \circ p = n \circ q$ and set up a rule defined by the relation $(m, q) \odot (n, r) = (m \circ n, q \circ r)$, then 1) C will be a K-class, 2) $\odot$ a C-rule, 3) C an abelian group as regards the rule denoted by the sign $\odot$.

Proof. 1) Either $m \circ p = n \circ q$ or $m \circ p$ is not $= n \circ q$ by hypothesis and § 1. Hence either $(m, q) = (n, p)$ or (m, q) is not $= (n, p)$. Obviously the assertions $(n, p) = (m, q)$ and $(m, q) = (n, p)$ are identical in meaning, since this is so for $n \circ q$ and $m \circ p$. It remains to verify the euclidean postulate.

Suppose that $(m, q) = (l, r)$ and $(l, r) = (n, p)$.
Then $m \circ r = l \circ q$ and $l \circ p = n \circ r$.
Hence $(m \circ r) \circ (l \circ p) = (n \circ r) \circ (l \circ q)$.

But since G is an abelian semigroup $(m \circ r) \circ (l \circ p) = m \circ r \circ l \circ p = (m \circ p) \circ (l \circ r)$ and likewise $(n \circ r) \circ (l \circ q) = (n \circ q) \circ (l \circ r)$. Therefore $(m \circ p) \circ (l \circ r) = (n \circ q) \circ (l \circ r)$. Whence $m \circ p = n \circ q$ by definition of semigroup and finally $(m, q) = (n, p)$ by definition of equality.

2) Let $(m', q') = (m, q)$ and $(n', r') = (n, r)$. Therefore $m' \circ q = m \circ q'$ and $n' \circ r = n \circ r'$. Whence as under 1) $(m' \circ n') \circ (q \circ r) = (m \circ n) \circ (q' \circ r')$ and hence $(m' \circ n', q' \circ r' = (m \circ n, q \circ r)$. That is $(m', q') \odot$

$(n', r') = (m, q) \odot (n, r)$, or equals by equals give equals and $\odot$ denotes a C-rule.

3) The fundamental group property is secured by the definition. To prove the associative law we have $(m, n) \odot \{(p, q) \odot (r, s)\} = (m \circ p \circ r, n \circ q \circ s) = (m, n) \odot (p, q) \odot (r, s)$. Suppose that $(m', q') \odot (n, r) = (m, q) \odot (n, r)$. Then $(m' \circ n, q' \circ r') = (m \circ n, q \circ r)$ by definition. Hence $(m' \circ n) \circ (q \circ r) = (m \circ n) \circ (q' \circ r)$ and $(m' \circ q) \circ (n \circ r) = (m \circ q') \circ (n \circ r)$. Therefore $m' \circ q = m \circ q'$ by § 14. That is $(m', q') = (m, q)$ and similarly for the other form of this property. Thus C is a semigroup. But the element (m, m) is a modulus and to every (n, q) corresponds an inverse (q, n). Therefore by *XIX*, C is a group, which we will denote by G.

Finally $(n, r) \odot (m, q) = (n \circ m, r \circ q)$
$= (m \circ n, q \circ r) = (m, q) \odot (n, r)$.

Therefore by § 16, G is an abelian group.

52. The semigroup C and group G are connected by *Proposition XXI.* If we declare $(m \circ q, q) = m$ then $(\underline{m \circ q, q}) \odot (\underline{n \circ r}, r) = m \circ n$. For $(m \circ q, q) \odot (n \circ r, r) = (m \circ q \circ n \circ r, q \circ r) = m \circ n \circ q \circ r, q \circ r) = m \circ n$.

53. *Scholium.* The group G contains a semigroup K holoedrically isomorphic with C in such manner that the rule $\odot$ becomes identical with the rule $\circ$ for K.

54. Obviously *Prop. XX* may be applied to the set of natural numbers in two different ways according as we use the semigroup on addition or that on multiplication. An essential distinction between these two procedures is furnished by

55. *Proposition XXII.* Given two rules of combination of which one is distributive over the other, no set of symbols can form a group with respect to both rules. For such a set would contain a modulus v for the lower rule (*XVI*). Then by the distributive law

$a = bx = b(x \circ v) = bx \circ bv = a \circ bv$ for every a. That is bv is also a modulus. But there can not be more than one modulus (*III*). Therefore $bv = v$ for every b so that when b' is not $= b$ we have $b'v = v = bv$ contrary to definition.

56. *Corollary.* It is not possible to define a set of symbols constituting a semigroup with modulus as regards both rules of § 55.

57. If *Prop. XX* is applied to the set of natural numbers and the rule of addition, we lose the semigroup on multiplication and can not recover it. This semigroup, in fact, is destroyed if we only annex a modulus for addition. Consequently we give the preference to the semigroup on multiplication and proceed with the group G obtained by applying *XX* to it.

58. *Proposition XXIII.* Given a higher rule distributive over a lower, and a set of symbols forming an abelian semigroup with respect to each rule, if we build the group G of § 52 with reference to the higher rule, then a necessary and sufficient condition of having a lower rule over which the higher shall be distributive and with regard to which G shall constitute an abelian semigroup is the formula of definition $(m, q) \circ (n, r) = (mr \circ nq, qr)$.

Proof. The necessity of this relation is obvious. To show that it is sufficient we first let $(m', q') = (m, q)$ and $(n', r') = (n, r) \therefore (m'q = mq')$ and $n'r = nr'$. Therefore $m'r'qr = m'qrr' = mrq'r'$ and $n'q'qr = nqq'r'$. Hence $(m'r' \circ n'q)qr = (mr \circ nq)q'r'$ by distributive law and $\therefore$ by definition $(m'r' \circ n'q'q'r') = (mr \circ nq, qr)$ or equals with equals give equals and we have a C-rule. Similarly equals with unequals give unequals. For the associative law $(k, l) \circ \{(m, q) \circ (n, r)\} = (k, l) \circ (mr \circ nq, qr) = (kqr \circ lmr \circ lnq, lqr) = (kq \circ lm, lq) \circ (n, r) = (k, l) \circ (m, q) \circ (n, r)$.

Obviously $(n, r) \circ (m, q) = (m, q) \circ (n, r)$. Thus we have an abelian semigroup and the distributive principle follows by taking equals with equals.

59. *Definition.* The symbols of G shall be ordered according to the convention that (m, q) shall precede or follow (n, q) according as m precedes or follows n and by § 3.

60. *Scholium.* If a, b denote two natural numbers that have no common factor other than unity, the set of symbols (a, b) is simply ordered by the above definition. This set, which we will denote by R, may be taken as representative of G since every element of G is equal to some member of R.

61. *Definition.* The symbols constituting the set R are called absolute, rational numbers.

62. *Proposition XXIV.* Every two unequal elements of G are connected by a relation $h = g \circ x$ where h, g denote the given elements, g being that which precedes.

For $g = (m, q) = (mr, qr)$ and $h = (n, r) = (nq, qr)$ and mr precedes nq by hypothesis. Therefore $h = g \circ x$ is satisfied by $x = (d, qr)$ where $nq = mr \circ d$.

63. *Corollary.* The set R is simply ordered according to the lower rule denoted by $\circ$.

64. *Proposition XXV.* If we exclude the modulus u the remaining symbols of R form a set which is simply ordered according to the higher rule. For let x denote any element of R which precedes and y any one which follows u.

Then gx precedes and gy follows g. That is, the trios u, g, gx and u, g, gy are ordered whether g precedes or follows u.

65. *Corollary.* The modulus u separates the remaining symbols of R into two simply ordered classes, neither of which contains either a first or last element.

66. It is possible to divide R into two ordered classes so that every element of R belongs either to the one or to the other, but neither class has either a first or a last member. For example, the class R_1 of all elements whose squares precede 2 and the class R_2 of all whose squares follow 2 together exhaust the set R, while neither R_1 nor R_2 has either a first or a last element. If we take 4 instead of 2, the element 2 is not included either in R or in R_2; it separates them. We usually make this separation in practice by selecting a well ordered set, *e. g.* according to the decimal scale. We take first the highest integer in R_1, then the highest number of tenths, hundredths and so on. Similarly we pick out first the lowest integer in R_2, then the lowest set of tenths, hundredths, etc.

67. *Definition.* A well ordered set which has no last element is called a series.

68. *Definition.* The series of all symbols of the well ordered set $1, 2, \ldots w, w+1, \ldots 2w, \ldots 3w, \ldots w^2, \ldots$ which precede the element a taken in the above order, is said to be a series of type a.

69. *Definition.* An ordered set which has no first nor last element will be called an unbounded set.

70. Given an unbounded set S ordered according to a rule with respect to which S constitutes an abelian semigroup, suppose that a series $K[f_n]$ of type w has been selected from among the elements of S and let W_1 denote the set of all symbols of S which follow, V_2 all which precede every f_i, let V_1 denote the set of all symbols s which precede W_1 and W_2 the set of all that follow V_2.

Then one of the sets W_1, V_1 must be unbounded and both may be. So also of V_2 and W_2.

71. In every one of the four cases when
W_1 has a first element or V_1 has a last
V_2 has a last element or W_2 a first
this element is called the limit of the set to which it belongs.

72. In case W_1 and V_1 or V_2 and W_2 are both unbounded we will introduce a new symbol f_w which we will call the limit of $K[f_n]$.

73. *Lemma.* If W_i and V_i are both unbounded for the series $K[f_n]$ the same is true for the series $K[a \circ f_n]$ where a is any symbol in S. For if either the W or V set corresponding to $K[a \circ f_n]$ had a limit l we could write $l = a \circ x$ and x would be a limit for the same W or V belonging to $K[f_n]$ contrary to hypothesis.

74. *Theorem.* If the series $K[f_i]$, $K[g_j]$ both divide S into two unbounded sets the same is true of the set $K[f_i \circ g_i]$.

Proof. Let $h_1 = f_1 \circ g_1$, $h_2 = f_2 \circ g_2$, ...
$h_{w+1} = f_2 \circ g_1$, $h_{w+2} = f_2 \circ g_2$, . . .
$h_{2w+1} = f_3 \circ g_1$, $h_{2w+2} = f_3 \circ g_2$, . . .

Then by the Lemma each of the series $K[h_j]$, $K[h_{w+j}]$, $K[h_{2w+j}]$, . . . $K[h_{iw+j}]$, . . . divides S into two unbounded sets. Hence the same is true of the set h_1, h_{w+2}, h_{2w+3}, . . . h_{iw+i+1}.

75. On the basis of the preceding theorem and definitions is built up, by laying down the usual definitions of equality, order and two rules of combination, the set X of limits of series $K[f_n]$ selected out of R. The set X forms an abelian group with reference to the higher rule of combination and an abelian semigroup on the lower rule. The higher rule is distributive over the lower and possesses the modulus u. X may be simply ordered according to the lower rule and is divided by the modulus u into two unbounded sets each

simply ordered according to the higher rule. X is itself an unbounded set and contains a sub set holoedrically isomorphic with R as regards each rule of combination. Thus the whole set of absolute numbers is deduced from the set of absolute rational numbers by using only principles of order.

76. The preceding development does not take account of the possibility that $K[f_n]$ may not really divide S. There may be no symbol which follows every f_i. Then W_1 is an empty class and V_1 is taken to coincide with S. Thus if $K[f_n]$ and $K[g_n]$ are both series of this kind, they both have the same W and V, whence if new symbols f_w, g_w were to be introduced the principle of definition used in § 75 would lead us to declare $f_w = g_w$. Therefore we assign to the totality of the sets $K[f_n]$ for which W_1 is an empty class a single symbol Z which is to follow every element of X. Then following § 75, $Z \circ g_w$ is to be defined as h_w where $h_n = f_n \circ g_n$ and $f_w = Z$. But for $K[h_w]$, W_1 is empty, therefore $Z \circ x = Z \circ g_w = h_w = Z$. That is, $Z + x = Z = x + Z$, $Zx = Z = xZ$, and similarly $Z + Z = Z = ZZ = Z^3 = Z^N$. The associative, commutative and distributive laws still hold as well as the first group property, but the semigroup is destroyed for both rules by violating the law of equals with unequals.

77. It is also possible that no symbol in S precedes every f_i. Then V_2 is an empty class and W_2 coincides with S. On the same principle as in § 76 we assign to the totality of series of this kind a single symbol v to precede every x and define $v \circ x = h_w$ where $h_n = f_n \circ g_n$, $g_w = x$, $f_w = v$. And now the two rules must be distinguished. Since no x precedes every f_i the sets $K[f_n + g_n]$ and $K[g_n]$ separate S into the same V and W. Therefore $v + x = h_w = g_w = x + v$. And for the same reason no s can precede every $f_n\, g_n$

$$\therefore \quad vx = (fg)_w = v = xv$$

Also in like manner $\mathrm{v} + \mathrm{v} = \mathrm{v} = \mathrm{vv}$.

In the set K [h, v] the associative, commutative and distributive laws and the first group property are all preserved. The law of equals with unequals is violated for multiplication but still holds for addition, for which v is a modulus.

78. Since the introduction of infinity would destroy both semigroups while zero destroys only that on multiplication, it seems preferable to admit zero to our arithmetic and exclude infinity. But this removes the only objection to enlarging the addition semigroup into a group. This is effected at a stroke by applying *Prop. XX* to the set *X* and thus building an abelian group with reference to addition. In this group we then define a higher rule by the formulas

$$\bar{a} \odot b = \overline{ab} = a \odot \bar{b} \text{ and } \bar{a} \odot \bar{b} = ab.$$

This completes the system of real numbers, forming an abelian group on addition and, when we exclude its modulus, an abelian group on multiplication, which is distributive over addition. The system of real numbers can be simply ordered according to the lower rule, but the abelian group on the higher rule can not be similarly treated.

79. We might have proceeded by first applying *XX* to the natural numbers as a semigroup on addition. This yields the whole set of integers, positive, negative and zero. Omitting zero and defining by the law of signs we should have an abelian semigroup on multiplication. Applying *XX* to it, there results an abelian group composed of all rational numbers except zero. Then the introduction of the limits would supply the whole set of irrational numbers and close by reintroducing zero.

80. As long as we postulate two rules of combination, one distributive over the other, and demand semi-

groups with reference to both, we are led inevitably to the transfinite types in case we impose no restriction on the multiplication table and, if we require a modulus, to the natural numbers, whence the system of real numbers results from the attempt to build groups. The two rules of combination connected by the distributive property may be regarded as defining arithmetic up to date; it has not yet been found profitable to postulate in any other way. Then transfinite arithmetic is distinguished by postulating the semigroups and finite arithmetic by the postulate of the modulus.

81. The real numbers form the most general finite system with a single unit and must enter into every system with more than one principal unit, one unit must always be the modulus. Systems with two or more principal units have been studied exhaustively by Weierstrass. Here the abelian group on the lower rule is postulated universally.

With two principal units, if we exclude the modulus of the lower rule, the necessary and sufficient condition of an abelian semigroup on the higher rule is the existence in the system of a symbol i such that $ii = \bar{u}$. Thus the common complex numbers form the most general system with two principal units satisfying the postulates for real members.

82. Within the system of common algebra are distinguished different number bodies each built on a root of a given algebraic equation as a unit. The algebraic numbers of a given body form an abelian group on addition and, excluding zero, an abelian group on multiplication, and must contain a given symbol, the root of the given algebraic equation. For example the system of common complex numbers is the number body which contains a root of the quadratic $x^2 + 1 = o$.

83. An integral algebraic number satisfies an equation

$$x^N + a_1 x^{N-1} + \dots + a_i x^{N-i} + \dots + a_n = 0,$$

where every a_i is an integer. A good illustration of the serviceableness of the group theory process for arithmetic is the theorem: The whole numbers of a quadratic number body form an abelian group on addition and, if we exclude zero, an abelian semigroup with regard to multiplication.

84. A more striking illustration is furnished by Dedekinds' "Ideals." An ideal is defined as a combination of the whole numbers of a number body possessing the first group property for both addition and multiplication. The ideals of a given body form an abelian semigroup with respect to multiplication. There is no addition of ideals.

85. Similar applications are found in the expression of an ideal by aid of its basis and in the cognate formulation of the transformations of a collineation in space of N dimensions. The latter question resolves itself into that of complex numbers with N principal units. Here it is no longer possible to preserve even the restricted abelian semigroup on the higher rule.

Not only is the abelian character lost, as in quaternions, but the semigroup property may be violated on account of the possibility of a combination by the higher rule being zero when none of the factors is zero.

86. It is now possible to describe more concisely the relation of the transfinite arithmetic to common arithmetic. The symbols defined in § 22 form a set of numbers with infinitely many principal units w, w^2, w^3, . . *i. e.*, the principal units form a series of type w. In finite arithmetic, it is true, the symbols u^2, u^3, . . . all belong to the set generated from u as principal unit, but if we allow this analogy in the transfinite system there will be still more new symbols.

For just as the removal of restrictions on the higher rule carried our series of symbols beyond the finite set of type w so the introduction of a still higher rule, corresponding to involution, will extend it beyond the set defined in § 22 if we impose no restrictions on the new rule except that of order and the special form of the group property.

87. Let us postulate a rule expressed by a^b higher than the two preceding, that is, the combination a b shall precede a^b. To distinguish we will now call ab the lower and a ○ b the lowest rule.

The lower rule has been shown to be associative and commutative for the set K [w □], which forms an abelian semigroup with regard to it. A set of symbols shall be generated from w by the higher rule in accordance with the postulate of order and the restricted form of the group property for w^a, where a is a previously defined element. By Prop. I this set forms a series w, $w^w = w'$, $w^{w'} = w''$, $w^{w''} w'''$, . . . and now w' must follow every w^N (N = 2, 3, . . .) [w^N is not a combination by rule, the N is a mere index]. Then by § 22 we obtain a series holoedrically isomorphic with the set there considered if to the lowest and lower rules respectively we make the lower and higher correspond.

88. The articulation of the two lower rules in §§ 28–35 leaves room for much freedom in definition. There we defined the lowest rule first and made the lower depend upon it. Here we have already defined the lower rule to the extent implied in the application of § 22. This, however, involves no further restriction than the inductive associative formula of § 19. But the definition from the lowest rule carries this property with it.

It is therefore permissible to define the lowest rule so that the lower shall be distributive over it. Then by the process of § 22 new combinations are made by the higher and lowest rule together, from each series of type w in the present set of symbols a new set of the same type as that in § 22. And all new series being fitted in by the nature of the process between consecutive elements of the series previously defined we have at every stage a totality of symbols forming a series.

89. The group properties of the two lowest rules are preserved throughout the preceding process. On every new symbol not obtained from the preceding by postulating them is built up a set forming abelian semigroups on both the lower and lowest rules, the new symbol playing the part of a unit precisely like w in the series of § 22, and in this set the lower rule is distributive over the lowest. This process is to be continued as long as new symbols can be obtained by it.

90. There remains one more possibility of building new symbols by introducing a combination defined one way of the higher symbols with those of the series K $[u \circ]$ according to the lowest rule. Thus to every a in the higher series will be assigned a w-series a, $\mathrm{a} \circ u$, $\mathrm{a} \circ u'$, $\mathrm{a} \circ u''$, $\mathrm{a} \circ u'''$, . . .

This series fits in between a and its next following element in the former set so that we still have a totality forming a series.

91. For completeness it is convenient also to define combinations one way of certain symbols with those of K $[u \circ]$ according to the other two rules, viz.: those expressions in which the symbols of K $[u \circ]$ have already been used as marks or indices, *e. g.*:

$u'w$, $u''w$, $u'''w$, . . . $w^{u'}$, $w^{u''}$, $w^{u'''}$, . . .
$u'w'$, $u''w'$, $u'''w'$, . . . $w'^{u'}$ $w'^{u''}$, $w'^{u'''}$, . . .

Expressions $u \circ a$, $u' \circ a$, . . . u'^{w}, u''^{w}, . . . $u'^{w'}$, $u''^{w'}$, . . . are not defined, neither are there any combinations according to the higher rule except the set w^{a} where a belongs to this set. The higher rule has not been made a rule of combination for any whole set. This would involve assigning properties to it and completing the arithmetic of the symbols we have defined, which is beyond the present purpose. It is enough to have them in series; that forms the foundation of their arithmetic.

92. The mere existence of the series of symbols developed in the preceding §§ is enough to solve a great variety of problems arising in analysis, the nature of which will be illustrated by comparing this series with series consisting of absolute numbers. The natural numbers form a series of type w if they are arranged in the order of their genesis. It is, however, possible to arrange them in series of higher types for *e. g.* the odd numbers alone form a series of type w. If to this we add on the even numbers successively we obtain series of types

$$w+1,\ w+2,\ \ldots\quad w+N,\ \ldots$$

and thus the whole set is ordered in type $2w$.

If we order the set R of § 60 as follows:

$$(1,1),\ (2,1),\ (3,1).\ (4,1),\ \ldots$$
$$(1,2),\ (3,2),\ (5,2),\ (7,2),\ \ldots$$
$$(1,3),\ (2,3),\ (5,3),\ (7,3),\ \ldots$$

we have a series which can be put ordinally into one to one correspondence with the series $1, 2, \ldots$ $w+1,\ w+2,\ \ldots\ 2w,\ 2w+1,\ \ldots$ *i. e.*, is of type w^2.

The same set of numbers (the common fractions) can, however, be arranged in a series of higher type. For every finite, simple, continued fraction is equal to some common fraction and conversely. Now let us order the finite continued fractions according to the values of their successive quotients

$$q_1,\ q_2,\ q_3,\ \ldots\ q_N.$$

Thus the continued fractions containing each a single quotient form a series of type w. Then from each member of this series by adding a second quotient results again type w. Therefore the fractions of the form $\frac{1}{q_1+}\frac{1}{q_2}$ may be arranged in series ordinally similar to 1, 2, . . . $w+1$, . . . $2w$, . . . $3w$, . . . *i. e.* of type w^2. Annexing a third quotient replaces each element of this set by a series of type w so that we include types w^2+1, w^2+2, . . . $2w^2$, . . . $3w^2$, . . . *i. e.* the class of fractions $\frac{1}{q_1+}\frac{1}{q_2+}\frac{1}{q_3}$ is ordered in type w^3.

Therefore the whole set of simple continued fractions, comprehending types w, . . . w^2, . . . w^3, . . . w^N, . . . constitutes, as ordered, a series of type $w^w = w'$.

93. Now it is easy to arrange the natural numbers also in a series of type w' as follows. We know that the class of prime numbers can be put into one to one correspondence with the whole set of natural numbers (the number of primes is infinite).

Therefore we can set up a one to one correspondence ordinally between the prime numbers and the simple continued fractions with a single quotient. By the same reasoning the continued fractions with two quotients are shown to be ordinally similar to the class of product of two primes, *i. e.*, if to every quotient q_1 we assign that prime p_1 for which q_1 is the ordinal number in sequence (so that *e. g.* to the quotient 7 we assign 17) and likewise for a second prime p_2 and quotient q_2, and so on.

Proceeding in this way the class of all products of N primes is exhibited as ordinally similar to the class of continued fractions $\frac{1}{q_1+}\frac{1}{q_2+}\cdots\frac{1}{q_N}$. But the class of all products of primes is the whole set of natural numbers. Therefore by § 92 the natural numbers may be arranged in series w'. Q. E. D.

94. In other words we have here a method whereby

the elements of any w-series may be rearranged so as to produce a series of type w^w. Applying this method successively to the w-series in the preceding we obtain in place of w, $2w$, . . . w', $2w'$ . . . that is, instead of w^2 we get a type ww'. Then w^2+w, w^2+2w, . . . are replaced by $ww'+w'$, $ww'+2w'$, . . . making in all type $2ww'$ replacing the former $2w^2$.

Then come in succession types $3ww'$, $4ww'$, . . . so that the original w^3 expands into a type w^2w'. It is clear that in this way we retrace precisely the steps of the process of § 22, building up on each new symbol the abelian semigroups according to the lower and lowest rules.

Therefore by successive applications of the method it will be possible to rearrange the natural numbers in series of types as high as any of the set hitherto defined. This result may also be stated. The transfinite ordinal series so far defined may each be put into one to one correspondence with the set of natural numbers.

95. The symbols w, $w+1$, $w+2$, . . . *i. e.*, the transfinite symbols, are said to form the second ordinal class, the finite symbols constituting the first class. The latter class was said to be of type w, where w is the symbol following next after all the finite symbols of $K[u\circ]$. Likewise we shall say that the first and second ordinal classes together form a series of type W, introducing this new symbol without assigning to it any properties except that it shall follow next after all symbols of the second ordinal class, *i. e.*, after all the set $K[w^a]$: w, w', w'', . . .

96. We see that the ordinal symbols play a double role. They were defined as symbols forming a well ordered set. But to each symbol which has no immediate predecessor corresponds a series of which it is the type, viz.: the series of all its predecessors or any ordinally similar series. The arrangements of the nat-

ural numbers in § 94 are only a part of these series. But they are also only a part of the permutations of the whole set of natural numbers. Other permutations are obtained by the following *Lemma.* From any w series of symbols may be obtained a set of permutations of the symbols forming a series of type $w' = w^w$.

Proof. Let $a_1, a_2, \ldots a_N, \ldots$ denote the given symbols in the given order. Without changing the order of the higher a's put a_1 successively in every subsequent place. This set of permutations is obviously of type w.

From every one of them by repeating the process on a_2 we obtain again a w-series ∴ in all a series of permutations of type w^2. Repeating the process with a_3, $a_4, \ldots$ successively we have a series of permutations whose type is w'. Q. E. D.

97. Thus the arrangements of the natural numbers furnish a series of type not lower than W. For by § 96 we first deduce from the series 1, 2, 3, ... *i. e.*, the normal order, a set of permutations forming a series of type w'. That is, to every ordinal symbol preceding w' is assigned a permutation, and *vice versa.* Then by § 94 arrange the natural numbers in series w' which obviously is a different permutation from any of the preceding ∵ it can not be obtained by the process of § 96. Now repeating the method of § 96 on each w-series of this permutation we use up all ordinal symbols between w' and that which results from it by a second application of § 94. This holds step by step as long as the latter process can be carried on. Thus to every ordinal symbol in succession preceding W is assigned a new permutation. That is, we have a set of permutations of all the natural numbers forming a series whose ordinal type is W.

98. The process just described for making permutations of all the natural numbers can not yield a series

of type higher than W since, as we have seen, it generates precisely the series W of ordinal symbols. That there are types higher than W is obvious, for we can proceed with W just as with w to generate new e-sets and new abelian semigroups, and there is no limit to the possibilities in the way of still higher symbols and rules. Now it is quite conceivable that there may be further permutations of natural numbers not obtainable by the above process. As a first step toward investigating this question let us consider the simpler one whether the natural numbers themselves can be arranged in a series of ordinal type higher than W. For this purpose we will establish the following

Lemma. Permutations of the natural numbers can be made by the process described and ordered so as to form a series of type higher than any series of all the natural numbers, however arranged. For by the process in question we can always form a new permutation which differs from the first permutation in at least its first element, from the second in at least its second element . . . . . . from the $(w+1)$st in at least its $(w+1)$st element, and so on, therefore is not included in any set of permutations ordinally similar to any possible arrangement of all the natural numbers. From this Lemma we readily obtain the

Theorem. Every possible arrangement of the natural numbers is a series of the second class. For by the Lemma to every such arrangement in series can be assigned a set of permutations forming a series of higher ordinal type. But by § 97 the process by which this is effected yields a set of permutations forming a series of type W. Therefore every possible arrangement of natural numbers in series is of type lower than W, therefore belongs to the second ordinal class (being *ipso facto* $\geq w$). Q. E. D.

99. The arrangement of the natural numbers in series of type higher than w finds application in the study of infinite continued fractions. By aid of the euclidean algorithm for greatest common divisor every absolute irrational number less than unity can be expressed as an infinite continued fraction

$$\frac{1}{q_1+}\,\frac{1}{q_2+}\;\cdots\cdots\;\frac{1}{q_N+}\;\cdots\cdots$$

and conversely. The class of infinite simple continued fractions may therefore be taken as the representative of the class of irrational numbers between zero and 1.

An infinite continued fraction can not be obtained from a finite one merely by annexing quotients; it can only be described by assigning a law which determines q_N for every value of N. An infinite continued fraction may be formed, *e. g.*, by the law that every quotient shall be 2. It is sufficient, however, to consider the class in which the quotients are all different; this can be put into one to one correspondence with the whole class. Accordingly we are concerned with the class of all possible permutations of all the natural numbers. These permutations may be examined in the same way as the permutations of a finite set by imagining a framework of places to be filled, but the number of places is infinite. Moreover, we must provide for the possibility of filling the places in a series of order higher than w. Thus an infinite continued fraction can be formed by filling the even places successively with the odd numbers in their natural order and the odd numbered places with even numbers in the same way, *i. e.*,

$$\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{3+}\,\frac{1}{6+}\,\frac{1}{5+}\;\cdots\cdots$$

or by filling the odd numbered places with primes and the even places with composite numbers. Or we can select first the places whose indices are primes, then the indices which are products of two primes, three primes, and so on, and fill these sets with the corresponding sets of numbers permuted in any way we please.

100. We have seen that every possible arrangement of all the natural numbers in a series is of type lower than W. Therefore the quotient places of our framework *in the order in which we propose to fill them* (or rather state how they shall be filled) constitute an ordinal type of the second class. That is, the *numbered places* in a permutation of all the natural numbers, arranged, however, *in the order in which they are to be filled*, form a series obtainable by the process of § 97.* Hence there can be no permutation of all the natural numbers not obtainable by this process, since such a permutation would be *eo ipso* different from all of those hitherto obtained and well ordered in some definite numbered places. In other words, this would mean that the series of the places as they are filled, or the series of numbers with which each place is to be filled, was not obtainable by the process of § 97. Therefore all possible permutations of the natural numbers form a series closely related to the series of type W.

BIOGRAPHY.

I, ARTHUR BOWES FRIZELL, member of the Protestant Episcopal Church, was born in Boston, July 14, 1865. My parents were Joseph Palmer Fessenden Frizell, civil and hydraulic engineer, and Julia Anna (Bowes) Frizell. My early education was chiefly at home and in a private school in Dorchester, Mass. I graduated from the high school in St. Paul, Minn., and spent three years at the Massachusetts Institute of Technology, where I afterwards served as assistant instructor in mathematics, 1888–'91. I received the degree of Bachelor of Arts from Harvard College in 1893 and that of Master of Arts from Harvard University in 1900. I served as instructor in mathematics at New York University 1895–'96, and at Harvard 1897–1906, when I resigned to study abroad. After three semesters at Göttingen, I returned to America, and was appointed, 1908, Professor of Mathematics in Midland College, Atchison, which position I resigned, 1909, to accept an instructorship in the University of Kansas.

*And the preceding statements apply *verbatim* to every series of values admissible in a given numbered place.